Algorithms and Data Structures with Applications in Machine Learning

The Stable Matching Problem and The Gale-Shapley algorithm

December 8, 2024

Outline

Introducing the Stable Matching problem

The Gale-Shapley Algorithm

Optimality of The Gale-Shapley Algorithm

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Bipartite Graph:

- A bipartite graph $G = (U, V, E)$ consists of:
  - Two disjoint sets of vertices: U and V.
  - Edges E: Connect vertices in U to vertices in V, representing potential pairings.

The Stable Matching Problem: A Graph Perspective

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- A bipartite graph $G = (U, V, E)$ consists of:
  - Two disjoint sets of vertices: U and V.
  - Edges E: Connect vertices in U to vertices in V, representing potential pairings.

Objective:

- Find a matching between U and V such that:
  - Each vertex is matched to at most one vertex from the other set.
  - The matching satisfies a property called **stability** (to be defined later).

Examples of applications

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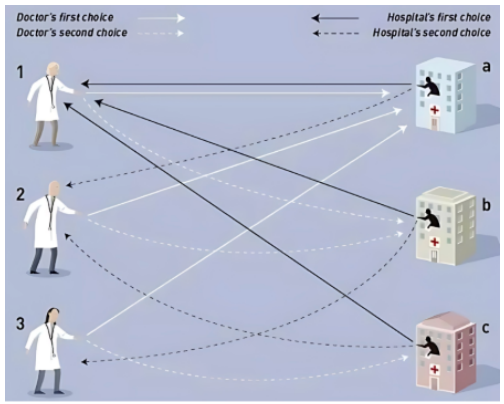
Applications:

- Assigning students to universities (e.g., Parcoursup in France).

- Matching medical residents to hospitals.



Vocabulary and a Philosophical Caveat

Vocabulary:

- To stay consistent with the original paper [1] by Gale and Shapley, we will use the terminology of "men" and "women" to describe the matching process.

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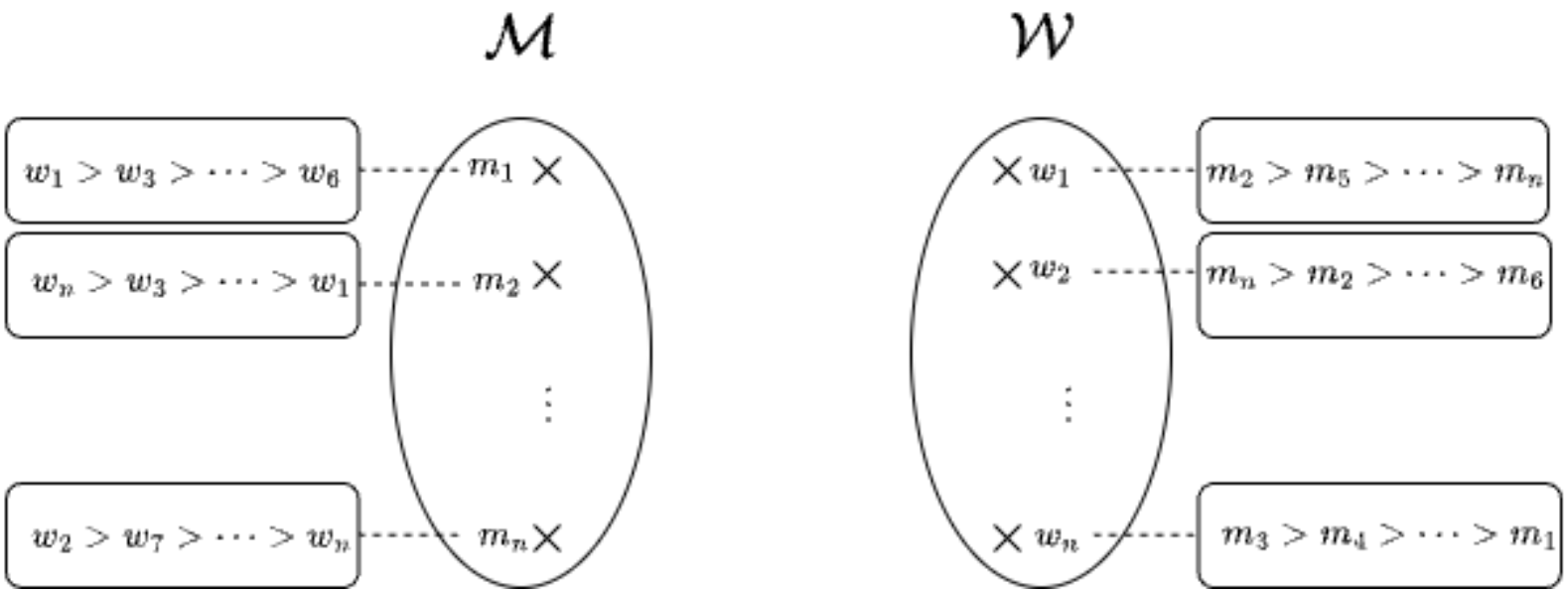
Philosophical Caveat:

- As Vladimir Jankélévitch reminds us in [2]: *'L'amour ne veut rien savoir sur ce qu'il aime ; ce qu'il aime, c'est le centre de la personne vivante, parce que cette personne est pour lui une fin en soi, ipséité incomparable, mystère unique au monde.'*
- Translation: "Love doesn't care to know what it loves; what it loves is the core of the living person, because this person is an end in itself, an incomparable selfhood, a unique mystery in the world."

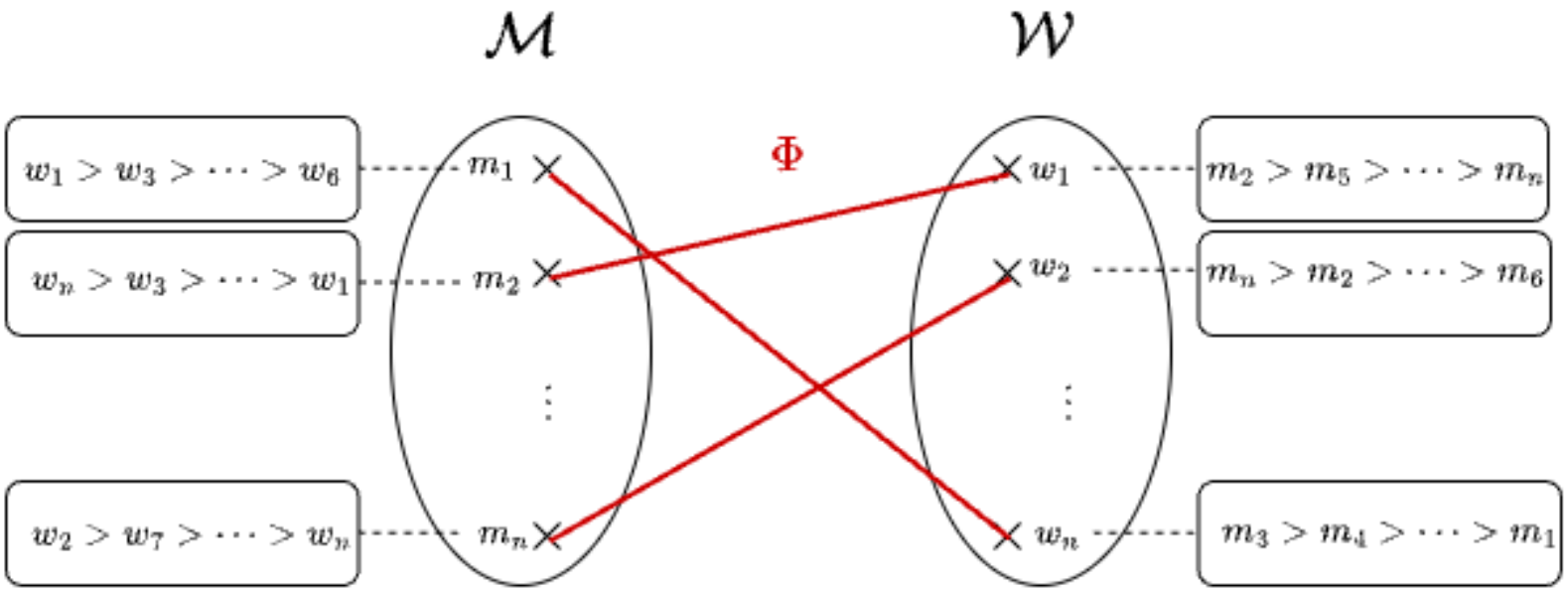
Introducing the data



An example

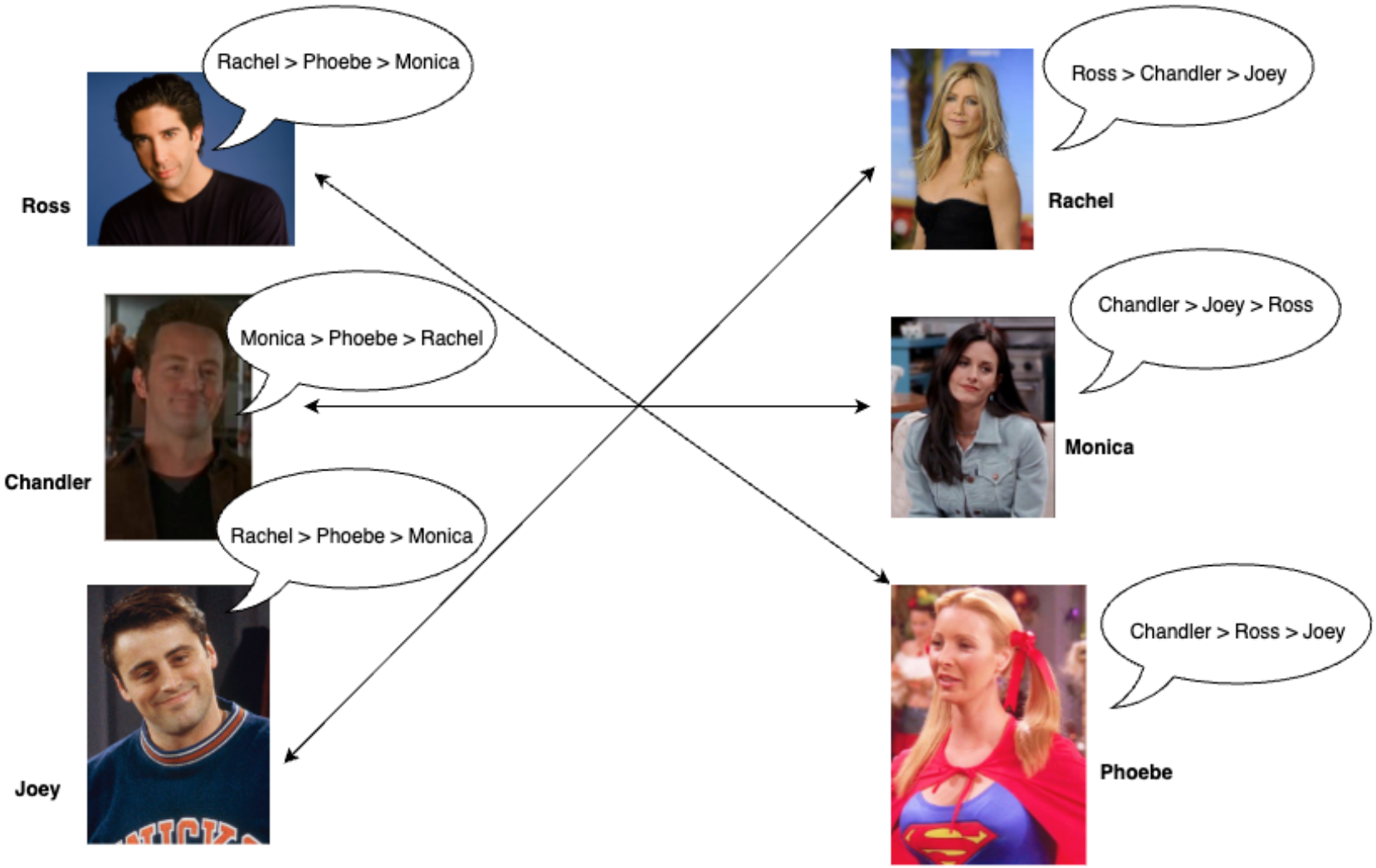

Rachel > Phoebe > Monica
Ross
Monica > Phoebe > Rachel
Chandler
Rachel > Phoebe > Monica
Joey
Ross > Chandler > Joey
Rachel
Chandler > Joey > Ross
Monica
Chandler > Ross > Joey
Phoebe

Defining a Matching



The resulting couples: $\{(m, \Phi(m)), m \in \mathcal{M}\}$

An example of a Matching

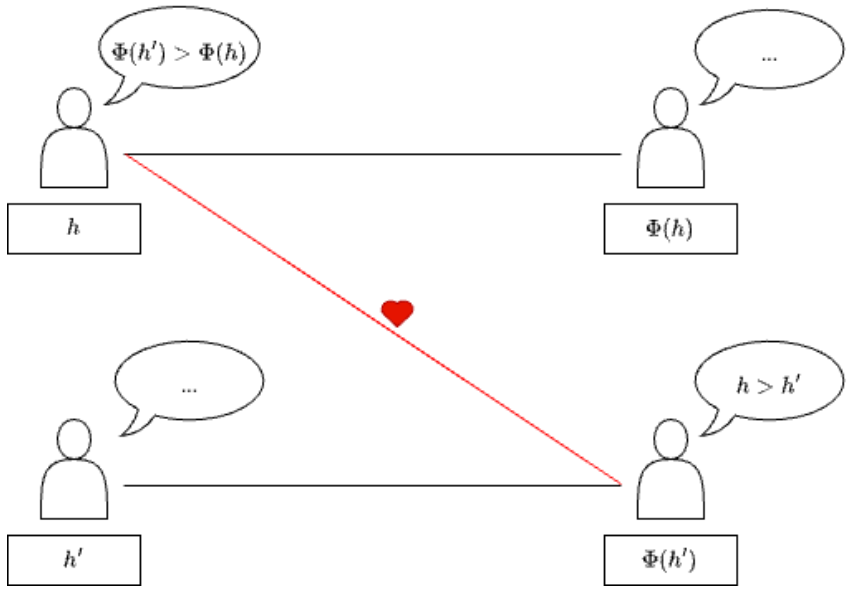


Definition of an instability

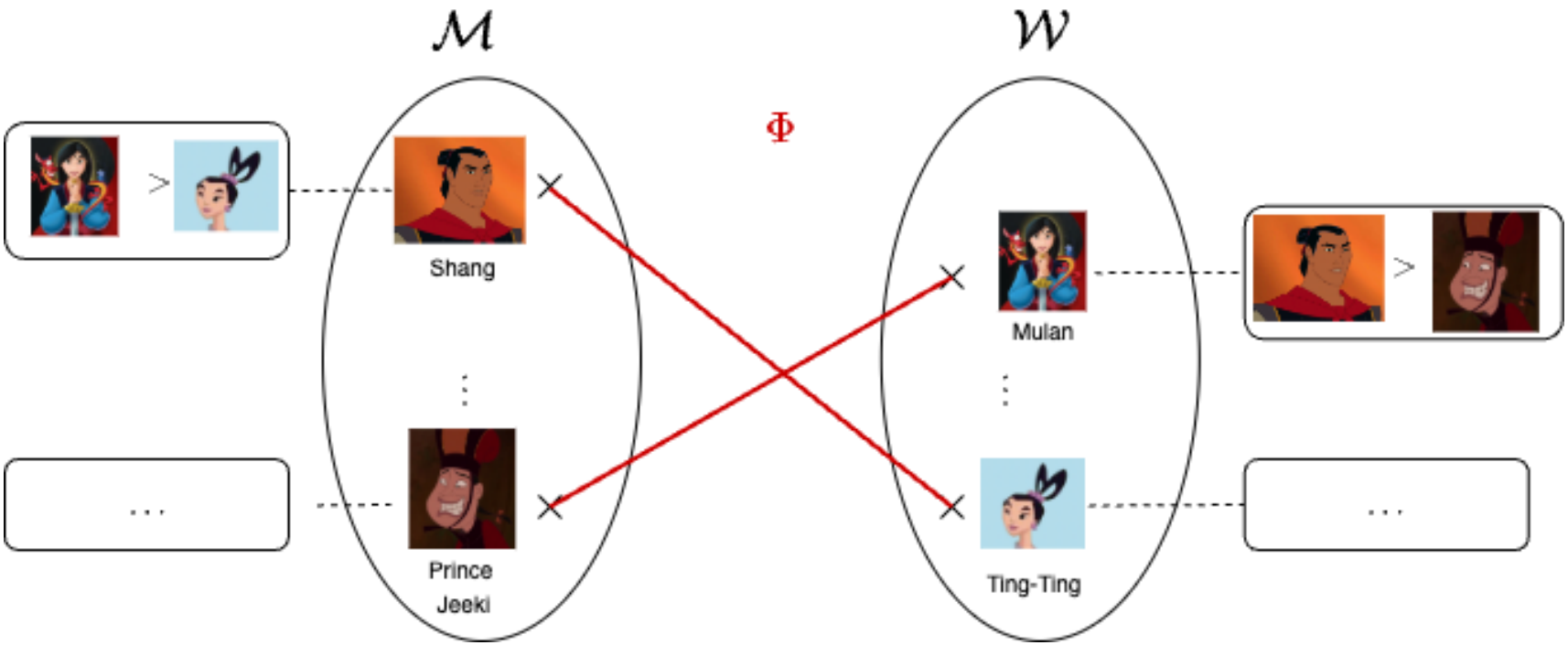
Definition

A matching is said to be unstable if there exists a pair of individuals who would prefer to be matched with each other over their current partners.

An example:



An example of an instability

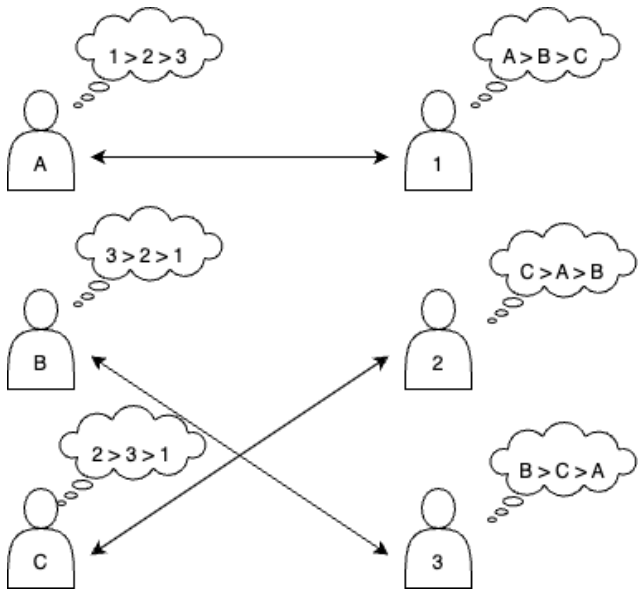


Defining a Stable Matching

Definition

A matching $\phi : \mathcal{H} \to \mathcal{W}$ is said to be **stable** if there is no instability within the pairs $\{m, \phi(m), m \in \mathcal{M}\}$.

An example:

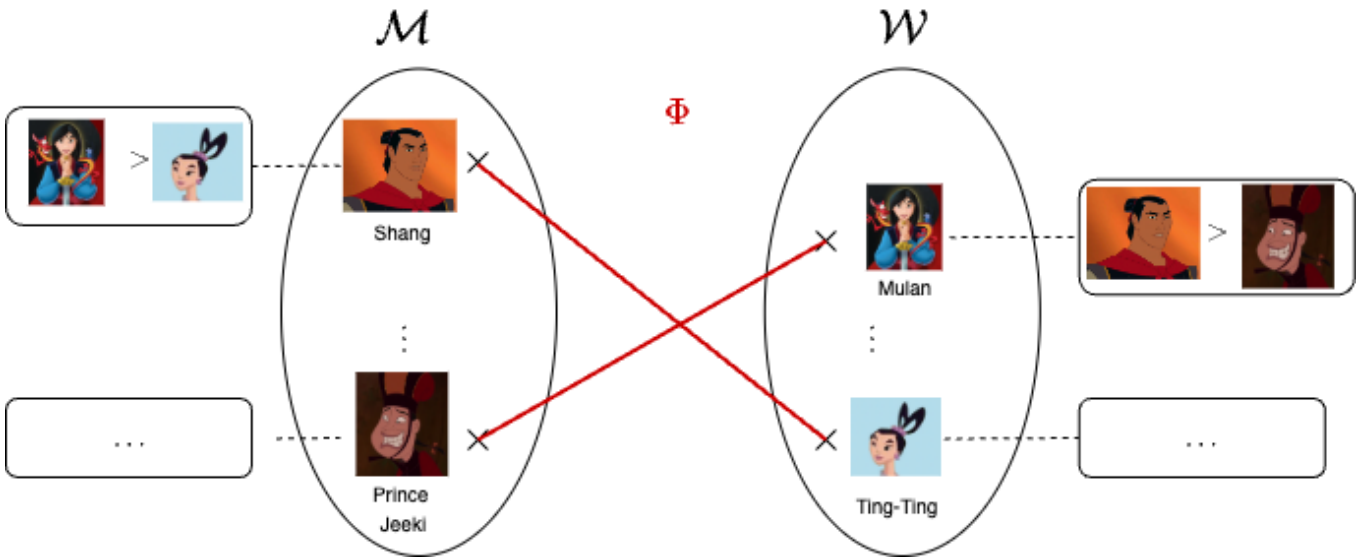


How Do We Build Stable Matchings?

Question: How do we construct a matching that satisfies the **stability** property?

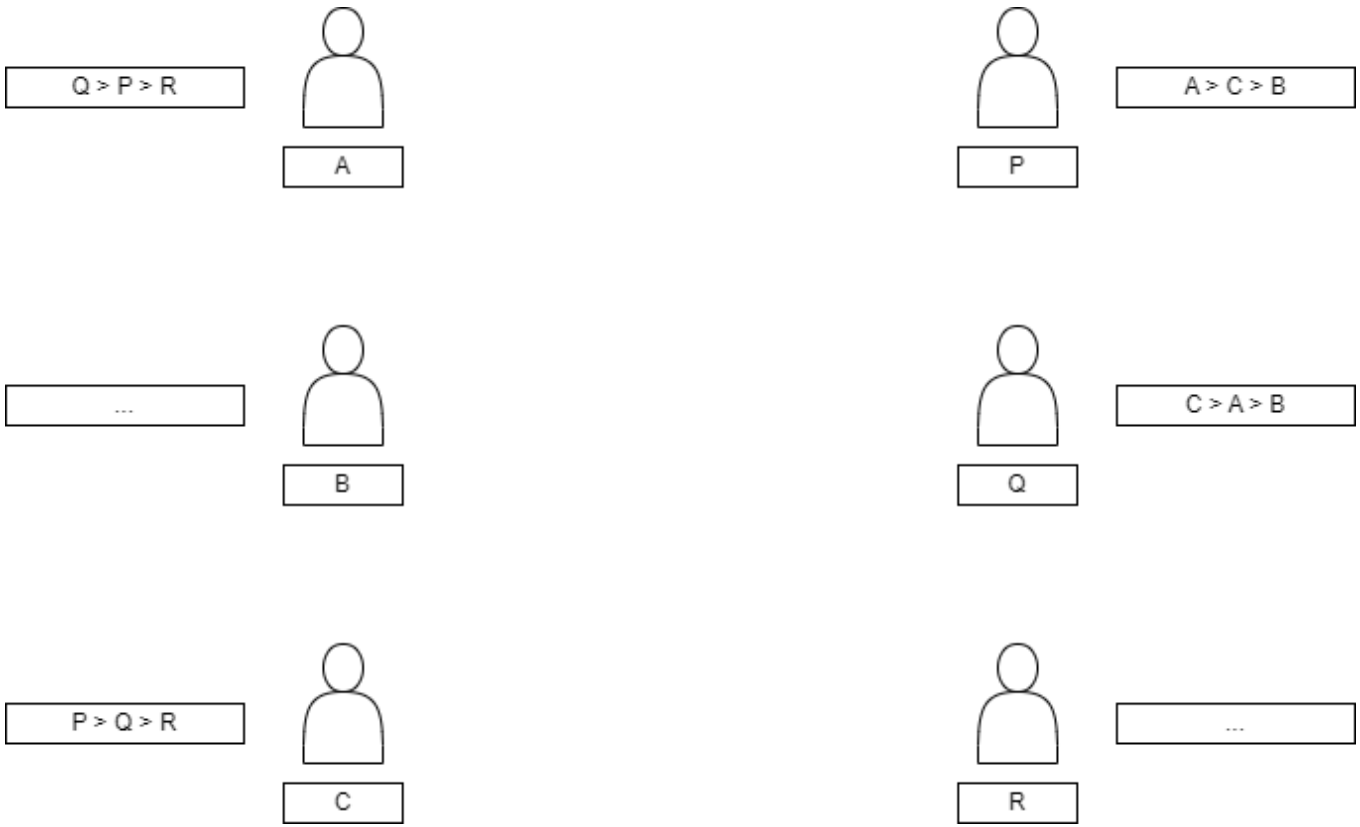
Naive Approach: What if we take a *laissez-faire* approach and resolve every instability **iteratively**?

For instance:



How Do We Build Stable Matchings?

- Iteratively addressing each instability might seem promising.
- However, could this process guarantee a **stable matching** in all cases?



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Algorithm Gale-Shapley Algorithm

Input: Lists of preferences (men, women)
Output: Stable matching

All people start as free
while $\exists$ free man m who hasn't proposed to all **do**
    Pick such a man m
    Let w be next woman on m's list
    if w is free **then**
        Engage m and w
    else if w prefers m to current m' **then**
        Engage m and w, free m'
    else
        w rejects m
    end if
end while

The Gale-Shapley Algorithm: an Example

Ross	R > P > M	Rachel	R > C > J
Chandler	R > M > P	Phoebe	C > R > J
Joey	R > M > P	Monica	C > J > R

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References

[1] David Gale and Lloyd S Shapley. "College admissions and the stability of marriage". In: *The American Mathematical Monthly* 69.1 (1962), pp. 9–15.

[2] Vladimir Jankélévitch and Béatrice Berlowitz. "Quelque part dans l'inachevé". In: (1978).

Thank you for your attention